

ON A CONSTRUCTION OF $C^1(\mathbb{Z}_p)$ FUNCTIONALS FROM \mathbb{Z}_p -EXTENSIONS OF ALGEBRAIC NUMBER FIELDS

TIMOTHY ALL AND BRADLEY WALLER

ABSTRACT. Let k be any number field and k_∞/k any \mathbb{Z}_p -extension. We construct a natural $\Lambda = \mathbb{Z}_p[[T-1]]$ -morphism from $\varprojlim k_n^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p$ into a special subset of $C^1(\mathbb{Z}_p)^*$, the collection of linear functionals on the set of continuously differentiable functions from $\mathbb{Z}_p \rightarrow \mathbb{C}_p$. We apply the results to the problem of interpolating Gauss sums attached to Dirichlet characters and the explicit annihilation of real ideal classes.

1. INTRODUCTION

Fix an odd prime p and let m be a positive integer co-prime to p . For an integer n , we let $\zeta_n = e^{2\pi i/n}$ so that $\zeta_n^d = \zeta_{n/d}$ for every $d \mid n$. Let $K_n = \mathbb{Q}(\zeta_{mp^{n+1}})$, and let $G_n = \text{Gal}(K_n/K_0)$.

We take a moment to review some classical theory from which this paper draws inspiration. Let $\theta_n \in \mathbb{Q}[\text{Gal}(K_n/\mathbb{Q})]$ denote the classical Stickelberger element attached to the number field K_n . Recall that θ_n , once properly made integral, annihilates the class group of K_n . Suppose φ is a non-trivial even Dirichlet character of conductor mp^{n+1} taking values in K , a finite extension of \mathbb{Q}_p . The character φ decomposes uniquely into a product of a tame character χ and a wild character ψ . Let $\theta_n(\chi) \in K[G_n]$ denote the χ -part of θ_n . In a celebrated work [?], Iwasawa showed that the sequence $(\theta_n(\chi)) \in \varprojlim K[G_n]$ (the projective limit taken with respect to the natural maps) is associated in a natural way to a function $F_\chi(T) \in \mathfrak{o}[[T-1]]$ where \mathfrak{o} is the integer ring of K . What's more, this function is essentially the p -adic L -function of Leopoldt and Kubota. In fact, we have

$$L_p(s, \chi\psi) = F_\chi(\zeta_\psi(1+p)^s)$$

where $\zeta_\psi = \overline{\psi}(1+p)$.

Unfortunately, if one restricts the action of θ_n to K_n^+ , the maximal real subfield of K_n , it reduces to a multiple of the norm. With \log_p denoting the Iwasawa logarithm, non-trivial explicit elements such as

$$\vartheta_n = \sum_{\sigma \in G(K_n/\mathbb{Q})} \log_p(1 - \zeta_{mp^{n+1}}^\sigma) \sigma^{-1}$$

Key words and phrases. distributions, L -functions, real abelian number field, class group.

were shown in [?], once properly made integral, to annihilate $\mathrm{Cl}(K_n^+) \otimes_{\mathbb{Z}} \mathcal{O}$ where \mathcal{O} is the ring of integers of the topological closure of $K_n^+ \hookrightarrow \mathbb{Q}_p^{\mathrm{alg}}$. This article was born out of considering what analytic functions were naturally associate to the sequences $(\vartheta_n(\chi)) \in \varprojlim \mathbb{Q}_p^{\mathrm{alg}}[G_n]$ (or more generally, to elements in $\varprojlim K_n^\times$) in analogy with Iwasawa's construction of p -adic L -functions.

Towards that end, let k be any number field, and let

$$k = k_0 \subset k_1 \subset k_2 \subset \cdots \subset \bigcup_{n=0}^{\infty} k_n = k_{\infty}$$

denote a \mathbb{Z}_p -extension of k . So $\Gamma := \mathrm{Gal}(k_{\infty}/k)$ is topologically isomorphic to \mathbb{Z}_p , and $\Gamma_n = \mathrm{Gal}(k_n/k_0) \simeq \Gamma/\Gamma^{p^n}$. Let γ_0 be a fixed topological generator for Γ and associate Γ with \mathbb{Z}_p via the isomorphism $\gamma_0^a \mapsto a$. Let \mathfrak{p} be a prime of k such that the inertia subgroup of \mathfrak{p} is $\mathrm{Gal}(k_{\infty}/k_i)$ for some i . This necessitates $\mathfrak{p} \mid p$. The valuation $v_{\mathfrak{p}}$ extends to k_{∞} , and we let \mathcal{K}_n denote the completion of k_n with respect to this valuation.

Let \mathbb{C}_p denote the topological closure of $\mathbb{Q}_p^{\mathrm{alg}}$. Suppose $\mu = \{\mu_n : \Gamma_n \rightarrow \mathbb{C}_p\}_{n=0}^{\infty}$ is a collection of maps with the following property:

$$\mu_n(x) = \sum_{y \mapsto x} \mu_{n+1}(y)$$

where $\Gamma_{n+1} \rightarrow \Gamma_n$ naturally. We call such a collection of maps a *distribution* on Γ . We denote the ring (under convolution) of all distributions on Γ by $\mathcal{D}(\Gamma)$, and we write $\mu(a + p^n \mathbb{Z}_p)$ in place of the more cumbersome $\mu_n(\gamma_0^a \bmod \Gamma^{p^n})$.

Note that the Γ -map $\mathcal{D}(\Gamma) \rightarrow \varprojlim \mathbb{C}_p[\Gamma_n]$ defined by

$$\mu \mapsto \left(\sum_{a=0}^{p^n-1} \mu(a + p^n \mathbb{Z}_p) \gamma_0^{-a} \right)$$

prescribes an isomorphism of rings. So the elements $(\vartheta_n(\chi)) \in \varprojlim \mathbb{Q}_p^{\mathrm{alg}}[G_n]$ naturally give rise to distributions in $\mathcal{D}(\Gamma)$ through the inverse of this map. On the other hand, $(\vartheta_n(\chi)) \in \varprojlim \mathbb{Q}_p^{\mathrm{alg}}[G_n]$ as a byproduct of $(1 - \zeta_{mp^{n+1}}) \in \varprojlim K_n^\times$, the projective limit taken with respect to the norm maps. Taken together, this uncovers a very natural source for distributions: define $\varprojlim K_n^\times \rightarrow \mathcal{D}(\Gamma)$ by $(\ell_n) \mapsto \lambda$ where

$$\lambda(a + p^n \mathbb{Z}_p) = -\log_p(\ell_n^{\gamma_0^a}) \in \mathcal{K}_n.^1$$

Let $\mathcal{M}(\Gamma)$ denote the collection of \mathbb{Z}_p -valued distributions, and let $\mathcal{K}(\Gamma)$ denote the $\mathcal{M}(\Gamma)$ -module generated by the image of the map described above.

¹Our choice of sign reflects the formula for $L_p(1, \chi)$.

What does one do with distributions anyway? For $\mu \in \mathcal{D}(\Gamma)$, we say that a function $f : \mathbb{Z}_p \rightarrow \mathbb{C}_p$ is μ -integrable to mean that the limit

$$\int_{\mathbb{Z}_p} f(x) d\mu := \lim_{n \rightarrow \infty} \sum_{a=0}^{p^n-1} f(a) \mu(a + p^n \mathbb{Z}_p)$$

exists. We call this limit the *Volkenborn integral of f with respect to μ* . The distinguishing feature of Volkenborn integration is the uniform choice of representatives from the classes $a + p^n \mathbb{Z}_p$ where $0 \leq a < p^n - 1$ (namely, the choosing of a itself).

Thus distributions give rise to linear functionals on appropriate function spaces. For example, it's well known that for every $\mu \in \mathcal{M}(\Gamma)$, the collection $C(\mathbb{Z}_p)$ of continuous functions on \mathbb{Z}_p are μ -integrable. So every $\mu \in \mathcal{M}(\Gamma)$ determines a linear functional on $C(\mathbb{Z}_p)$ where

$$\mu(f) := \int_{\mathbb{Z}_p} f d\mu.$$

What's more, the Fourier transform $\mathcal{M}(\Gamma) \rightarrow \Lambda := \mathbb{Z}_p[[T-1]]$ given by $\mu \mapsto \hat{\mu}(T)$ where

$$\hat{\mu}(T) = \mu(T^x) = \int_{\mathbb{Z}_p} T^x d\mu(x) = \sum_{m=0}^{\infty} \left(\int_{\mathbb{Z}_p} \binom{x}{m} d\mu(x) \right) (T-1)^m$$

is a well-defined isomorphism. All told we have natural isomorphisms

$$\begin{array}{ccc} \mathcal{M}(\Gamma) & \longrightarrow & \Lambda \\ & \searrow & \downarrow \\ & & \mathbb{Z}_p[[\Gamma]] \end{array} \quad \text{given by} \quad \begin{array}{ccc} \mu & \longmapsto & F_\mu \\ & \searrow & \downarrow \\ & & (\sum \mu(a + p^n \mathbb{Z}_p) \gamma_0^{-a}) \end{array}$$

If M is a module over $\mathcal{M}(\Gamma)$ or $\mathbb{Z}_p[[\Gamma]]$ naturally, then we consider it a module over Λ (or any of the others for that matter) through the above diagram. In particular, extend the Iwasawa logarithm \log_p to a function $\text{Log}_p : k_n^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p \rightarrow \mathbb{C}_p$ in the obvious way: $\text{Log}_p(\ell \otimes x) = x \log_p(\ell)$. Then it's straightforward to verify that the map $\varprojlim k_n^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p \rightarrow \mathcal{K}(\Gamma)$ defined by

$$\mathfrak{l}_n \mapsto \left(\mathfrak{L} : a + p^n \mathbb{Z}_p \mapsto \text{Log}_p(\mathfrak{l}_n^a) \right)$$

is a Λ -morphism.

Our main result is that continuously differentiable functions are λ -integrable for every $\lambda \in \mathcal{K}(\Gamma)$, in other words

Theorem. *Let $\lambda \in \mathcal{K}(\Gamma)$. Then λ is a linear functional on $C^1(\mathbb{Z}_p)$ where*

$$\lambda(f) = \int_{\mathbb{Z}_p} f d\lambda.$$

In particular, the Fourier transform $\widehat{\lambda}(T) \in \mathbb{C}_p[[T-1]]$ exists and has radius of convergence ≥ 1 . The analytic functions $\widehat{\lambda}(T)$ are like L -functions for the underlying norm coherent sequence. For example, consider the following special case. Suppose k/\mathbb{Q} is an abelian number field whose conductor is not divisible by p^2 , and let F be any abelian number field linearly disjoint from k and of conductor co-prime to p . If k_∞/k is the cyclotomic \mathbb{Z}_p -extension of k , then the tower of number fields Fk_n forms the cyclotomic \mathbb{Z}_p -extension of Fk , and we consider Γ_n (resp. $\Delta := \text{Gal}(k_0/\mathbb{Q})$) as being contained in (resp. a quotient of) the set of automorphisms of $\text{Gal}(Fk_n/\mathbb{Q})$ fixing F . For a character χ of Δ , define $\varprojlim (Fk_n)^\times \rightarrow \mathcal{D}(\Gamma)$ by $(\ell_n) \mapsto \lambda_\chi$ where

$$\lambda_\chi(a + p^n \mathbb{Z}_p) = - \sum_{\delta \in \Delta} \log_p(\ell_n^{\alpha_\delta}) \overline{\chi}(\delta).$$

Let $\mathcal{K}_\chi^F(\Gamma)$ denote the Λ -module generated by the image of the map described above. The functions $\widehat{\lambda}_\chi(T)$ (or $\widehat{\lambda}(T)$, for that matter) interpolate values reminiscent of those found in the formula for $L_p(1, \varphi)$, the p -adic L -function of Leopoldt, Kubota, Iwasawa, et al. As a straightforward consequence of the above theorem, we have

Theorem. *Let $\lambda_\chi \in \mathcal{K}_\chi^F(\Gamma)$. Then λ_χ is a linear functional on $C^1(\mathbb{Z}_p)$ where*

$$\lambda_\chi(f) = \int_{\mathbb{Z}_p} f \, d\lambda_\chi.$$

If ψ is a character of Γ_n with $\zeta_\psi = \overline{\psi}(\gamma_0)$ and $(\ell_n) \mapsto \lambda_\chi$, then

$$\widehat{\lambda}_\chi(\zeta_\psi) = - \sum_{\sigma} \log_p(\ell_n^\sigma) \overline{\varphi}(\sigma)$$

where the sum runs over all $\sigma \in \Gamma_n \times \Delta = \text{Gal}(k_n/\mathbb{Q})$ and $\varphi = \chi\psi$.

We apply the above results to the problem of interpolating Gauss sums attached to a Dirichlet character. Particularly interesting is the case when the tamely ramified character χ is of conductor p . In this case, the Gauss sums

$$\tau(\chi\psi) = \sum_{a=1}^{p^{n+1}} \chi\psi(a) \zeta_{p^{n+1}}^a$$

are essentially interpolated from the Fourier transform of $\lambda_\chi \in \mathcal{K}_\chi^{\mathbb{Q}(\zeta_{p-1})}(\Gamma)$ where the underlying norm coherent sequence generates the projective limit of principal units of $\mathbb{Q}_p(\zeta_{p^{n+1}})$. Since it's peripheral to the interpolation problem, we also show how to use the special values of the functions $\widehat{\lambda}_\chi(T)$ to construct an explicit sequence $(\vartheta_n) \in \mathbb{Z}_p[\Gamma_n]$ such that ϑ_n annihilates the χ -part of the Sylow p -subgroup of $\text{Cl}(\mathbb{Q}(\zeta_{p^{n+1}}^+))$ for every $n \geq 0$.

2. VOLKENBORN DISTRIBUTIONS

In this section we give an overview of the theory of *Volkenborn distributions* of which distributions in $\mathcal{K}(\Gamma)$ are a special case. C. Barbacioru [?] developed the Volkenborn distribution in his doctoral dissertation. This section is largely an overview of the tools from [?] that will be needed in the sequel.

Definition 2.1. A distribution μ , on \mathbb{Z}_p , is said to be Volkenborn if there exists $B(\mu) \in \mathbb{R}_{\geq 0}$ such that

$$|p\mu(a + p^{n+1}\mathbb{Z}_p) - \mu(a + p^n\mathbb{Z}_p)|_p \leq B(\mu)$$

for all $a \in \mathbb{Z}_p$ and $n \in \mathbb{Z}_{\geq 0}$

Note that all p -adically bounded distributions are necessarily Volkenborn, but a distribution need not be bounded to be Volkenborn. In fact, the prototype Volkenborn distribution is the Haar distribution: $a + p^n\mathbb{Z}_p \mapsto \frac{1}{p^n}$.

Lemma 2.2. Let μ be a Volkenborn distribution and let $f_n : \mathbb{Z}_p \rightarrow \mathbb{C}_p$ be defined by $f_n : x \mapsto p^n\mu(x + p^n\mathbb{Z}_p)$. Then there exists a continuous and bounded function $f : \mathbb{Z}_p \rightarrow \mathbb{C}_p$ such that $f_n \rightrightarrows f$ uniformly on \mathbb{Z}_p .

Proof. Note that

$$p^n\mu(a + p^n\mathbb{Z}_p) = \left(\sum_{j=1}^n p^{j-1}(p\mu(a + p^j\mathbb{Z}_p) - \mu(a + p^{j-1}\mathbb{Z}_p)) \right) + \mu(\mathbb{Z}_p).$$

The terms of the sum go to zero as $j \rightarrow \infty$ since μ is Volkenborn. It follows that the sum converges. Define $f : \mathbb{Z}_p \rightarrow \mathbb{C}_p$ by $x \mapsto \lim p^n\mu(x + p^n\mathbb{Z}_p)$. Note the above shows that f is bounded, in fact, $|f(x)| \leq \max\{B(\mu), \mu(\mathbb{Z}_p)\}$ for all $x \in \mathbb{Z}_p$.

Now, let $x \in \mathbb{Z}_p$ be arbitrary. Let $m > n$ be sufficiently large so that

$$\begin{aligned} |f(x) - f_n(x)|_p &\leq \max \left\{ \{|f_{j+1}(x) - f_j(x)|_p\}_{j=n}^{m-1} \cup \{|f(x) - f_m(x)|_p\} \right\} \\ &\leq \max \{|f_{j+1}(x) - f_j(x)|_p\}_{j=n}^{m-1} \\ &\leq \frac{B(\mu)}{p^n}. \end{aligned}$$

The above bound does not depend on x , so $f_n \rightrightarrows f$. The function f is continuous since it is a uniform limit of continuous functions on a compact set. \square

For a Volkenborn distribution μ , we want to show that all C^1 functions are μ -integrable. The strategy will be to first show that polynomials are μ -integrable. This, in conjunction with properties of Mahler series of C^1 functions, will give us the μ -integrability of C^1 functions.

Proposition 2.3. Let μ be a Volkenborn distribution and P be a polynomial. Then P is μ -integrable.

Proof. Since limits are finitely additive, it suffices to show that $P(x) = x^m$ is μ -integrable for all $m \in \mathbb{Z}_{\geq 0}$. We proceed by induction. For $P(x) = 1$, we have

$$\int_{\mathbb{Z}_p} d\mu = \lim_{n \rightarrow \infty} \sum_{a=0}^{p^n-1} \mu(a + p^n \mathbb{Z}_p) = \mu(\mathbb{Z}_p).$$

Now, let $S_{n,m} := \sum_{j=0}^{p^n-1} j^m \mu(j + p^n \mathbb{Z}_p)$ for $m, n \in \mathbb{Z}_{\geq 0}$. We wish to show that for a fixed $m \geq 1$ that $S_{n,m}$ is a Cauchy sequence. Note that

$$\begin{aligned} S_{n+1,m} - S_{n,m} &= \sum_{j=0}^{p^{n+1}-1} \sum_{k=0}^{p-1} ((j + kp^n)^m - j^m) \mu(j + kp^n + p^{n+1} \mathbb{Z}_p) \\ (\star) \quad &= \sum_{j=0}^{p^{n+1}-1} \sum_{k=0}^{p-1} \sum_{l=1}^m \binom{m}{l} (kp^n)^l j^{m-l} \mu(j + kp^n + p^{n+1} \mathbb{Z}_p). \end{aligned}$$

By Lemma 2.2 we only need to show that the $l = 1$ term from (\star) is small. To do so, we will rewrite that term as follows:

$$\sum_{j=0}^{p^{n+1}-1} \sum_{k=0}^{p-1} m k p^n j^{m-1} \mu(j + kp^n + p^{n+1} \mathbb{Z}_p) = a_n + b_n$$

where

$$\begin{aligned} a_n &= \sum_{j=0}^{p^{n+1}-1} \sum_{k=0}^{p-1} m k p^n j^{m-1} (\mu(j + kp^n + p^{n+1} \mathbb{Z}_p) - \frac{1}{p} \mu(j + p^n \mathbb{Z}_p)), \\ b_n &= \sum_{j=0}^{p^{n+1}-1} \sum_{k=0}^{p-1} m k p^{n-1} j^{m-1} \mu(j + p^n \mathbb{Z}_p). \end{aligned}$$

It remains to show that both a_n and b_n go to zero as $n \rightarrow \infty$. For a_n , we have

$$\begin{aligned} |a_n|_p &= \left| \sum_{j=0}^{p^{n+1}-1} \sum_{k=0}^{p-1} m k p^n j^{m-1} (\mu(j + kp^n + p^{n+1} \mathbb{Z}_p) - \frac{1}{p} \mu(j + p^n \mathbb{Z}_p)) \right|_p \\ &\leq |m p^n (\mu(j + kp^n + p^{n+1} \mathbb{Z}_p) - \frac{1}{p} \mu(j + p^n \mathbb{Z}_p))|_p \\ &\leq p^{1-n} B(\mu). \end{aligned}$$

It follows that $a_n \rightarrow 0$ as $n \rightarrow \infty$. For b_n , we have

$$b_n = \sum_{j=0}^{p^{n+1}-1} \sum_{k=0}^{p-1} m k p^{n-1} j^{m-1} \mu(j + p^n \mathbb{Z}_p) = \frac{p-1}{2} m p^n S_{n,m-1}.$$

By the inductive hypothesis $\{S_{n,m-1}\}_{n=0}^{\infty}$ is a bounded sequence (since it is a convergent sequence). It follows that $b_n \rightarrow 0$ as $n \rightarrow \infty$. This shows that $S_{n,m}$ is a Cauchy sequence, so $\lim_{n \rightarrow \infty} S_{n,m}$ converges. \square

Since C^1 functions are determined by their Mahler series, it is important to know bounds on $\left| \int_{\mathbb{Z}_p} \binom{x}{m} d\mu(x) \right|$. The next proposition gives such a bound.

Proposition 2.4. *Let μ be a Volkenborn distribution. Then there exists $c \in \mathbb{R}_{\geq 0}$ such that for all $m \in \mathbb{Z}_{\geq 0}$ we have*

$$\left| \int_{\mathbb{Z}_p} \binom{x}{m} d\mu(x) \right|_p \leq cm$$

Proof. For $m = 0$, we know that $\int_{\mathbb{Z}_p} d\mu(x)$ exists and equals $\mu(\mathbb{Z}_p)$. From this point on let $m \in \mathbb{Z}_{\geq 1}$. By Proposition 2.3 we know that $\binom{x}{m}$ is μ integrable. The proof of the inequality proceeds in a similar manner to the proof of Proposition 2.3, and we will use the sequence $\{T_{n,m}\}_{n=0}^{\infty}$ where

$$T_{n,m} := \sum_{j=0}^{p^n-1} \binom{j}{m} \mu(j + p^n \mathbb{Z}_p).$$

Note that

(2.1)

$$|T_{n+1,m} - T_{n,m}|_p = \left| \sum_{j=0}^{p^{n+1}-1} \sum_{k=0}^{p-1} \left(\binom{j+kp^n}{m} - \binom{j}{m} \right) \mu(j+kp^n + p^{n+1} \mathbb{Z}_p) \right|_p.$$

To estimate Equation 2.1, we use the binomial identity

$$\binom{j+kp^n}{m} = \sum_{l=0}^m \binom{j}{l} \binom{kp^n}{m-l}.$$

The right hand side of Equation 2.1 becomes

$$(2.2) \quad \left| \sum_{j=0}^{p^{n+1}-1} \sum_{k=0}^{p-1} \sum_{l=0}^{m-1} \binom{j}{l} \binom{kp^n}{m-l} \mu(j+kp^n + p^{n+1} \mathbb{Z}_p) \right|_p.$$

We can bound each term of the sum from Equation 2.2 as follows:

$$\begin{aligned} \left| \binom{j}{l} \binom{kp^n}{m-l} \mu(j+kp^n + p^{n+1} \mathbb{Z}_p) \right|_p &\leq \left| \binom{kp^n}{m-l} \mu(j+kp^n + p^{n+1} \mathbb{Z}_p) \right|_p \\ &= \left| \frac{kp^n}{m-l} \binom{kp^n-1}{m-l-1} \mu(j+kp^n + p^{n+1} \mathbb{Z}_p) \right|_p \\ &\leq \left| \frac{kp^n}{m-l} \mu(j+kp^n + p^{n+1} \mathbb{Z}_p) \right|_p \\ &\leq p^{-n} m |\mu(j+kp^n + p^{n+1} \mathbb{Z}_p)|_p \\ &\leq Cpm \quad \text{by Lemma 2.2.} \end{aligned}$$

This estimate gives us that Equation 2.2 is bounded above by Cpm . In other words,

$$(2.3) \quad |T_{n+1,m} - T_{n,m}|_p \leq Cpm.$$

Now we are in position to prove the result.

$$\begin{aligned} |T_{n,m}|_p &= \left| \sum_{j=0}^n (T_{n,m} - T_{n-1,m}) + T_{0,m} \right|_p \\ &\leq \max\{Cpm, |T_{0,m}|\} \\ &= \max\{Cpm, |\mu(\mathbb{Z}_p)|_p\}. \end{aligned}$$

Letting $c = \max\{Cp, |\mu(\mathbb{Z}_p)|_p\}$, we see that $|T_{n,m}|_p \leq cm$. This gives us that $|\int_{\mathbb{Z}_p} \binom{x}{m} d\mu(x)|_p \leq cm$, as claimed. \square

It is important to note that c from Proposition 2.4 is independent of m .

Theorem 2.5. *Let $f \in C^1(\mathbb{Z}_p)$ and μ be a Volkenborn distribution. Then f is μ -integrable.*

Proof. Since $f \in C^1$, we know that the Mahler series of f is of the form

$$\sum_{m=0}^{\infty} a_m \binom{\cdot}{m} \quad \text{where} \quad \lim_{m \rightarrow \infty} m|a_m|_p = 0$$

(see [?]). We will show that

$$(2.4) \quad \int_{\mathbb{Z}_p} f(x) d\mu(x) = \sum_{m=0}^{\infty} a_m \int_{\mathbb{Z}_p} \binom{x}{m} d\mu(x).$$

By Proposition 2.4 we know that $|\int_{\mathbb{Z}_p} \binom{x}{m} d\mu(x)|_p \leq cm$. This tells us that

$$\lim_{m \rightarrow \infty} a_m \int_{\mathbb{Z}_p} \binom{x}{m} d\mu(x) = 0.$$

Thus the right hand side of Equation 2.4 converges.

Now we will show that the left hand side of Equation 2.4 exists and equals the right hand side of the same equation. To do so we will use the sequence $\{T_{n,m}\}_{m=0}^{\infty}$ from Proposition 2.4. The proof of Proposition 2.4 showed that there exists $c \in \mathbb{R}_{\geq 0}$ such that $|T_{n,m}|_p \leq cm$.

Let $\epsilon > 0$. Then there exists $M \in \mathbb{Z}_{>0}$ such that for all $m \geq M$ we have that $|a_m T_{n,m}|_p < \epsilon$. Also, there exists $N \in \mathbb{Z}_{>0}$ such that for all $0 \leq m \leq M$ and $n \geq N$ we have that $|a_m(T_{n,m} - \int_{\mathbb{Z}_p} \binom{x}{m} d\mu(x))|_p < \epsilon$.

Let $n \geq N$. Then

$$\sum_{j=0}^{p^n-1} f(j) \mu(j + p^n \mathbb{Z}_p) - \sum_{m=0}^{\infty} a_m \int_{\mathbb{Z}_p} \binom{x}{m} d\mu(x) = a_M + b_M$$

where

$$\begin{aligned} a_M &= \sum_{j=0}^{p^n-1} \sum_{m=0}^M a_m \binom{j}{m} \mu(j + p^n \mathbb{Z}_p) - \sum_{m=0}^M a_m \int_{\mathbb{Z}_p} \binom{x}{m} d\mu(x) \\ &= \sum_{m=0}^M a_m \left(T_{n,m} - \int_{\mathbb{Z}_p} \binom{x}{m} d\mu(x) \right) \end{aligned}$$

and

$$\begin{aligned} b_M &= \sum_{j=0}^{p^n-1} \sum_{m=M}^{\infty} a_m \binom{j}{m} \mu(j + p^n \mathbb{Z}_p) - \sum_{m=M}^{\infty} a_m \int_{\mathbb{Z}_p} \binom{x}{m} d\mu(x) \\ &= \sum_{m=M}^{\infty} a_m T_{n,m} - \sum_{m=M}^{\infty} a_m \int_{\mathbb{Z}_p} \binom{x}{m} d\mu(x). \end{aligned}$$

We have $|a_M|_p < \epsilon$ by our choice of n (which depends on M), and $|b_M|_p < \epsilon$ by our choice of M . It follows that $\int_{\mathbb{Z}_p} f(x) d\mu(x)$ exists, so f is μ -integrable. \square

3. THE MODULE OF VOLKENBORN DISTRIBUTIONS

Let $\mathcal{V}(\Gamma)$ denote the collection of Volkenborn distributions. Recall that $\Lambda \simeq \mathcal{M}(\Gamma)$ acts on $\mathcal{V}(\Gamma)$ by convolution.

Lemma 3.1. $\mathcal{V}(\Gamma)$ is a Λ -module.

Proof. Let $\nu \in \mathcal{V}(\Gamma)$ and μ a bounded distribution, i.e., a distribution such that there exists $B \in \mathbb{R}_{\geq 0}$ satisfying

$$|\mu(a + p^n \mathbb{Z}_p)|_p \leq B$$

for all a and n . We show more generally that $\nu * \mu \in \mathcal{V}(\Gamma)$. By the definition for convolution, we have

$$(\nu * \mu)(a + p^{n+1} \mathbb{Z}_p) = \sum_{j=0}^{p^n-1} \sum_{k=0}^{p-1} \nu(j + kp^n + p^{n+1} \mathbb{Z}_p) \mu(a - j - kp^n + p^{n+1} \mathbb{Z}_p),$$

and similarly

$$(\nu * \mu)(a + p^n \mathbb{Z}_p) = \sum_{j=0}^{p^n-1} \nu(j + p^n \mathbb{Z}_p) \sum_{k=0}^{p-1} \mu(a - j - kp^n + p^{n+1} \mathbb{Z}_p).$$

So we see that $p \cdot (\nu * \mu)(a + p^{n+1} \mathbb{Z}_p) - (\nu * \mu)(a + p^n \mathbb{Z}_p)$ equals

$$\sum_{j=0}^{p^n-1} \sum_{k=0}^{p-1} \mu(a - j - kp^n + p^{n+1} \mathbb{Z}_p) \cdot (p \cdot \nu(j + kp^n + p^{n+1} \mathbb{Z}_p) - \nu(j + p^n \mathbb{Z}_p)).$$

Since μ is bounded and

$$p \cdot \nu(j + kp^n + p^{n+1} \mathbb{Z}_p) - \nu(j + p^n \mathbb{Z}_p)$$

is bounded independently from j and k , we see that $\nu * \mu \in \mathcal{V}(\Gamma)$. \square

The Fourier transform of a Volkenborn distribution is guaranteed to exist from Theorem 2.5. We now study how convolution by $\mu \in \mathcal{M}(\Gamma)$ affects the Fourier transform of $\nu \in \mathcal{V}(\Gamma)$. For a Volkenborn distribution ν , let f_ν denote the function defined by $x \mapsto \lim p^n \nu(x + p^n \mathbb{Z}_p)$. Recall that f_ν is a bounded continuous function by Lemma 2.2. Let \mathbf{S} denote the indefinite-sum operator. For $f \in C(\mathbb{Z}_p)$, the action of \mathbf{S} on f simply shifts the Mahler expansion in the following way:

$$\mathbf{S}f = \mathbf{S} \sum_{m=0}^{\infty} \binom{\cdot}{m} (\nabla^m f)(0) = \sum_{m=0}^{\infty} \binom{\cdot}{m+1} (\nabla^m f)(0) \in C(\mathbb{Z}_p)$$

where $(\nabla f)(x) = f(x+1) - f(x)$ is the finite-difference operator. The reader should consult [?, Chapter V] for more details.

Proposition 3.2. *Let $\mu \in \mathcal{M}(\Gamma)$. For every $\nu \in \mathcal{V}(\Gamma)$, we have*

$$(\widehat{\nu * \mu})(T) = \widehat{\nu}(T) \cdot \widehat{\mu}(T) - \log_p(T) \cdot \sum_{m=0}^{\infty} \mu(\mathbf{S}^{m+1}(f_\nu \circ \iota)) (T-1)^m$$

where $\iota : x \mapsto -1 - x$ is the canonical involution of \mathbb{Z}_p .

Proof. Note that

$$\sum_{a,b=0}^{p^n-1} T^{a+b} \nu(a + p^n \mathbb{Z}_p) \mu(b + p^n \mathbb{Z}_p) \xrightarrow{n \rightarrow \infty} \int_{\mathbb{Z}_p} T^x d\nu(x) \cdot \int_{\mathbb{Z}_p} T^x d\mu(x).$$

Consider the sum on the left. Collecting all terms such that $a+b \equiv c \pmod{p^n}$, we see that it equals

$$\sum_{c=0}^{p^n-1} T^c (\mu * \nu)(c + p^n \mathbb{Z}_p) + \sum_{c=0}^{p^n-2} \sum_{d=1}^{p^n-c-1} (T^{c+p^n} - T^c) \mu(c+d + p^n \mathbb{Z}_p) \nu(-d + p^n \mathbb{Z}_p).$$

As $n \rightarrow \infty$, the term on the left converges to $(\widehat{\mu * \nu})(T)$ since $\mu * \nu$ is Volkenborn. Hence the term on the right converges. We rewrite that term as

$$\frac{T^{p^n} - 1}{p^n} \sum_{m=0}^{\infty} \left(\sum_{c=0}^{p^n-2} \binom{c}{m} \sum_{d=1}^{p^n-c-1} \mu(c+d + p^n \mathbb{Z}_p) \cdot p^n \nu(-d + p^n \mathbb{Z}_p) \right) (T-1)^m.$$

This expression converges to $\log_p(T) \cdot G(T)$ where the m -th coefficient of $G(T)$ equals

$$g_m := \lim_{n \rightarrow \infty} \sum_{c=0}^{p^n-2} \binom{c}{m} \sum_{d=1}^{p^n-c-1} \mu(c+d + p^n \mathbb{Z}_p) \cdot p^n \nu(-d + p^n \mathbb{Z}_p).$$

We collect terms according to $\mu(j + p^n \mathbb{Z}_p)$ obtaining

$$g_m = \lim_{n \rightarrow \infty} \sum_{j=0}^{p^n-1} \left(\binom{\cdot}{m} \otimes (f_n \circ \iota) \right) (j) \cdot \mu(j + p^n \mathbb{Z}_p)$$

where $f_n : x \mapsto p^n \nu(x + p^n \mathbb{Z}_p)$ and \otimes is the shifted-convolution product. We now use the fact that $\left(\begin{smallmatrix} \cdot \\ m \end{smallmatrix}\right) \otimes g = \mathbf{S}^{m+1} g$ and $f_n \rightrightarrows f_\nu$ to obtain

$$\begin{aligned} g_m &= \lim_{n \rightarrow \infty} \sum_{j=0}^{p^n-1} \mathbf{S}^{m+1}(f_\nu \circ \iota)(j) \mu(j + p^n \mathbb{Z}_p) \\ &= \int_{\mathbb{Z}_p} \mathbf{S}^{m+1}(f_\nu \circ \iota) \, d\mu(x). \end{aligned}$$

This completes the proof of the proposition. \square

Remark 3.3. Note that if $\nu \in \mathcal{M}(\Gamma)$, then $f_\nu \equiv 0$. So we recover the well-known fact that $\widehat{\nu * \mu} = \widehat{\nu} * \widehat{\mu}$, the Fourier transform of a convolution of measures equals the convolution of Fourier transforms.

We now show that $\mathcal{V}(\Gamma)$ is populated by members of $\mathcal{K}(\Gamma)$.

Theorem 3.4. $\mathcal{K}(\Gamma)$ is a sub-module of $\mathcal{V}(\Gamma)$.

Proof. In light of Lemma 3.1, it suffices to prove that the generators of $\mathcal{K}(\Gamma)$ reside in $\mathcal{V}(\Gamma)$. Let $(\ell_n) \in \varprojlim k_n^\times$ with associated distribution λ . We have

$$p\lambda(a + p^n \mathbb{Z}_p) - \lambda(a + p^{n-1} \mathbb{Z}_p) = \log_p \left(\frac{\ell_{n-1}^{\gamma_0^a}}{\ell_n^{p\gamma_0^a}} \right).$$

Observe that

$$\frac{\ell_{n-1}^{\gamma_0^a}}{\ell_n^{p\gamma_0^a}} \xrightarrow{N_{n-1}^n} 1$$

where N_{n-1}^n is the norm from k_n to k_{n-1} . Since k_n/k_{n-1} is a cyclic extension, Hilbert's Theorem 90 gives an element $\alpha_n \in k_n^\times$ such that

$$\frac{\ell_{n-1}^{\gamma_0^a}}{\ell_n^{p\gamma_0^a}} = \alpha_n^{\gamma_0^a(\gamma_n-1)} \quad \text{where} \quad \gamma_n = \gamma_0^{p^{n-1}}.$$

It remains to show that $\log_p(\alpha_n^{\gamma_0^a(\gamma_n-1)})$ is bounded independent of a and n . In fact, we need only show that it is bounded independent of a and n for all n sufficiently large.

Assume that the inertia subgroup for \mathfrak{p} of k is $\text{Gal}(k_\infty/k_i)$ and let $n \geq i$. Let π_n be a local parameter for \mathcal{K}_n . Since $\mathcal{K}_n/\mathcal{K}_i$ is totally ramified, it follows that

$$N_i^n(\pi_n) = \pi_i$$

is a local parameter for \mathcal{K}_i . Moreover, we get that

$$N_i^n(\mathcal{K}_n^\times) = \langle \pi_i \rangle \times N_i^n(U_n)$$

where U_n denotes the units of \mathcal{K}_n . Note that

$$[U_i : N_i^n(U_n)] = p^{n-i}$$

since $\mathcal{K}_n/\mathcal{K}_i$ is cyclic and totally ramified. Let $U_i^{(j)}$ denote the j -th group of principal units of \mathcal{K}_i , so $U_i^{(j)} = 1 + (\pi_i)^j \subset U_i$. Let q denote the order of the residue class field for \mathcal{K}_i , and recall the filtration

$$U_i \supset U_i^{(1)} \supset U_i^{(2)} \supset \dots$$

where

$$[U_i^{(j)} : U_i^{(j+1)}] = \begin{cases} q-1 & j=0 \\ q & \text{else.} \end{cases}$$

Let m be the smallest integer such that $U_i^{(m)} \subset N_i^n(U_n)$, so $\langle \pi_i \rangle \times U_i^{(m)} \subseteq N_i^n(\mathcal{K}_n^\times)$. From the above filtration, we see that as n increases so must m . Let n be large enough so that $m > 1$.

Since $\langle \pi_i \rangle \times U_i^{(m)} \subseteq N_i^n(\mathcal{K}_n^\times)$, local class field theory gives us that $\mathcal{K}_n \subseteq \mathcal{L}_m$ where \mathcal{L}_m is the field of π_i^m -division points of some Lubin-Tate module for π_i (see [?, ?]). For a real number $s \geq -1$, we define the s -th ramification group

$$G_s(\mathcal{L}_m/\mathcal{K}_i) = \{\sigma \in \text{Gal}(\mathcal{L}_m/\mathcal{K}_i) : w(\sigma(a) - a) \geq s+1 \quad \forall a \in \mathcal{O}\}$$

where \mathcal{O} is the valuation ring of \mathcal{L}_m and w is the valuation associate to its maximal ideal. The Lubin-Tate extensions have the property that

$$G_{q^{m-1}-1}(\mathcal{L}_m/\mathcal{K}_i) = \text{Gal}(\mathcal{L}_m/\mathcal{L}_{m-1}).$$

Let $H \subset \text{Gal}(\mathcal{L}_m/\mathcal{K}_i)$ such that \mathcal{K}_n is the fixed field of H . A theorem of Herbrand (see [?, II.10.7]) gives us that

$$G_s(\mathcal{L}_m/\mathcal{K}_i)H/H = G_t(\mathcal{K}_n/\mathcal{K}_i) \quad \text{where} \quad t = \int_0^s \frac{dx}{[G_0(\mathcal{L}_m/\mathcal{K}_n) : G_x(\mathcal{L}_m/\mathcal{K}_n)]}.$$

By the minimality of m , we have that

$$G_{q^{m-1}-1}(\mathcal{L}_m/\mathcal{K}_i) = \text{Gal}(\mathcal{L}_m/\mathcal{L}_{m-1}) \not\subseteq H,$$

so for $s = q^{m-1} - 1$, we have $G_t(\mathcal{K}_n/\mathcal{K}_i)$ is non-trivial. We now obtain a crude but functional lower bound for the value t . Since $\mathcal{L}_m/\mathcal{K}_i$ is totally ramified, we have

$$t = \frac{[\mathcal{K}_n : \mathcal{K}_i]}{[\mathcal{L}_m : \mathcal{K}_i]} \sum_{j=1}^{q^{m-1}-1} \#G_j(\mathcal{L}_m/\mathcal{K}_n) \geq \frac{p^{n-i}}{q-1} \cdot \frac{q^{m-1}-1}{q^{m-1}} \geq \frac{p^{n-i-1}}{q-1} = t(n)$$

where the last inequality follows because $m > 1$. It follows that

$$\gamma_n \in \text{Gal}(\mathcal{K}_n/\mathcal{K}_{n-1}) \subseteq G_t(\mathcal{K}_n/\mathcal{K}_i) \subseteq G_{t(n)}(\mathcal{K}_n/\mathcal{K}_i).$$

Let $e(\pi_n : p)$ denote the ramification index of π_n over p , then for all n sufficiently large

$$\alpha_n^{\gamma_0^a(\gamma_n-1)} - 1 \in (\pi_n)^{t(n)} \Rightarrow v_p(\alpha_n^{\gamma_0^a(\gamma_n-1)} - 1) \geq \frac{t(n)}{e(\pi_n : p)} = \frac{1}{p(q-1)e(\pi : p)}.$$

Whence $\log_p(\alpha_n^{\gamma_0^a(\gamma_n-1)})$ is bounded independent of n and a for all n sufficiently large. This proves the theorem. \square

Remark 3.5. Let $H^1(\mathbb{C}_p)$ denote the ring of power series in $\mathbb{C}_p[[T-1]]$ convergent on the open ball of radius 1 centered about 1. This section shows that the map $\mathcal{F} : \varprojlim k_n^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p \rightarrow H^1(\mathbb{C}_p)/(\log_p(T))$ defined by

$$(\mathfrak{l}_n) \mapsto \widehat{\mathfrak{L}} \bmod (\log_p(T))$$

is a Λ -morphism. If $(\mathfrak{l}_n) \in \ker \mathcal{F}$, then for every $n \geq 0$, for every character ψ of Γ_n , we have

$$0 = \widehat{\mathfrak{L}}(\zeta_\psi) = \sum_{a=0}^{p^n-1} \overline{\psi}(\gamma_0^a) \text{Log}_p(\mathfrak{l}_n^a)$$

where

$$e_\psi \cdot \sum_{a=0}^{p^n-1} \text{Log}_p(\mathfrak{l}_n^a) \gamma_0^{-a} = \sum_{a=0}^{p^n-1} \overline{\psi}(\gamma_0^a) \text{Log}_p(\mathfrak{l}_n^a) \cdot e_\psi \in \mathbb{C}_p[\Gamma_n]$$

and $e_\psi \in \mathbb{C}_p[\Gamma_n]$ is the idempotent associate to ψ . Since $\mathbb{C}_p[\Gamma_n] = \bigoplus_\psi \mathbb{C}_p e_\psi$, it follows that

$$\begin{aligned} \widehat{\mathfrak{L}} \equiv 0 \bmod (\log_p(T)) &\Leftrightarrow 0 = \sum_{a=0}^{p^n-1} \text{Log}_p(\mathfrak{l}_n^a) \gamma_0^{-a}, \quad \forall n \geq 0 \\ &\Leftrightarrow \mathfrak{L} = 0. \end{aligned}$$

Whether \mathfrak{L} is the 0-distribution is a more delicate question. For suppose $\mathfrak{l}_n = \sum(\ell_j \otimes x_j)$, then

$$\mathfrak{L}(p^n \mathbb{Z}_p) = \text{Log}_p(\mathfrak{l}_n) = \sum x_j \log_p(\ell_j).$$

We now need to know whether the terms $\log_p(\ell_j)$ are p -adically independent, a question related to Leopoldt's conjecture.

4. APPLICATIONS

In this section we apply the above results to the problem of interpolating Gauss sums. For a Dirichlet character φ , let f_φ denote the conductor of φ and let $\tau(\varphi)$ denote the Gauss sum

$$\tau(\varphi) = \sum_{a=1}^{f_\varphi} \varphi(a) \zeta_{f_\varphi}^a.$$

We associate Dirichlet characters of conductor dividing mp^{n+1} to characters of $\text{Gal}(\mathbb{Q}(\zeta_{mp^{n+1}})/\mathbb{Q})$ in the obvious way: $\varphi(\sigma_a) = \varphi(a)$ where $\sigma_a : \zeta_{mp^{n+1}} \mapsto \zeta_{mp^{n+1}}^a$.

Theorem 4.1. *Let $k = \mathbb{Q}(\zeta_{mp})$ and k_∞/k the cyclotomic \mathbb{Z}_p -extension of k . Let $F = \mathbb{Q}(\zeta_{p-1})$. Let χ be a character of Δ with $m \mid f_\chi$, and let λ_χ be*

the distribution associate to the norm coherent sequence $(\zeta_{p-1}^t - \zeta_{mp^{n+1}}) \in \varprojlim (Fk_n)^\times$. If ψ is a character of Γ_n with $\zeta_\psi = \overline{\psi}(\gamma_0)$, then

$$\widehat{\lambda_\chi}(\zeta_\psi) = -(1 - \overline{\varphi}(p)) \sum_{a=1}^{f_\varphi} \log_p(\zeta_{p-1}^t - \zeta_{f_\varphi}^a) \overline{\varphi}(a)$$

where $\varphi = \chi\psi$.

Proof. Note that $\zeta_{p-1}^t - \zeta_{mp^{n+1}}$ is indeed a norm coherent sequence by virtue of our choice for ζ_n , namely, $\zeta_n^d = \zeta_{n/d}$ for all $d \mid n$. So λ_χ is an honest distribution satisfying

$$\widehat{\lambda_\chi}(\zeta_\psi) = \int_{\mathbb{Z}_p} \overline{\psi}(\gamma_0)^x \, d\lambda_\chi(x) = \sum_{a=0}^{p^n-1} \overline{\psi}(\gamma_0^a) \lambda_\chi(a + p^n \mathbb{Z}_p)$$

where the last equality follows since ψ is a character of Γ_n . Hence

$$\begin{aligned} \widehat{\lambda_\chi}(\zeta_\psi) &= - \sum_{\substack{b=1 \\ (b, mp)=1}}^{mp^{n+1}} \log_p(\zeta_{p-1}^t - \zeta_{mp^{n+1}}^b) \overline{\varphi}(b) \\ &= -(1 - \overline{\varphi}(p)) \sum_{b=1}^{mp^{n+1}} \log_p(\zeta_{p-1}^t - \zeta_{mp^{n+1}}^b) \overline{\varphi}(b) \\ &= -(1 - \overline{\varphi}(p)) \sum_{b=1}^{f_\varphi} \log_p(\zeta_{p-1}^t - \zeta_{f_\varphi}^b) \overline{\varphi}(b), \end{aligned}$$

and the theorem follows. \square

Corollary 4.2. *If $p-1 \mid t$, then*

$$\widehat{\lambda_\chi}(\zeta_\psi) = \frac{1 - \overline{\varphi}(p)}{1 - \varphi(p)/p} \tau(\overline{\varphi}) L_p(1, \varphi).$$

where $L_p(s, \varphi)$ is the Leopoldt-Kubota p -adic L -function. In particular, if $p \mid f_\varphi$, then

$$\widehat{\lambda_\chi}(\zeta_\psi) = \tau(\overline{\varphi}) L_p(1, \varphi).$$

Proof. The first fact follows immediately from the formula (see [?])

$$L_p(1, \varphi) = - \left(1 - \frac{\varphi(p)}{p}\right) \frac{1}{\tau(\overline{\varphi})} \sum_{a=1}^{f_\varphi} \log_p(1 - \zeta_{f_\varphi}^a) \overline{\varphi}(a).$$

The second fact follows from the first since if $p \mid f_\varphi$, then $\varphi(p) = 0$. \square

Combining the above corollary with results from Iwasawa [?] allow us to view the Gauss sums $\tau(\overline{\chi\psi})$ in an interesting light when the conductor of $\chi\psi$ is a p -power. Essentially, they arise as special values of the Fourier transform of a generating sequence for the projective limit of principal units of $\mathbb{Q}_p(\zeta_{p^{n+1}})$.

Theorem 4.3. *Let $k = \mathbb{Q}(\zeta_p)$ and k_∞/k the cyclotomic \mathbb{Z}_p -extension of k . Let $F = \mathbb{Q}(\zeta_{p-1})$. Let χ be a non-trivial even character of Δ . There exists a $p-1$ -st root of unity $\zeta_\chi \neq 1$ such that the distribution $v_\chi \in \mathcal{K}_\chi^F(\Gamma)$ associate to the norm coherent sequence $(\zeta_\chi - \zeta_{p^{n+1}}) \in \varprojlim (Fk_n)^\times$ satisfies*

$$\widehat{v}_\chi(\zeta_\psi) = \tau(\overline{\chi\psi}) \cdot (\text{unit})$$

for every wildly ramified character ψ .

Proof. Let U_n denote the principal units of $\mathbb{Q}_p(\zeta_{p^{n+1}})$. Let $U = \varprojlim U_n$ where the projective limit is with respect to the norm maps, and let $U(\Gamma)$ denote the collection of distributions associate to the norm coherent sequences of U . U is naturally a $\mathbb{Z}_p[[\Gamma]]$ -module, $U(\Gamma) \subseteq \mathcal{D}(\Gamma)$ is naturally an $\mathcal{M}(\Gamma)$ -module, and they are, in fact, isomorphic Λ -modules.

Now, let $C_n \subseteq U_n$ denote the topological closure of the cyclotomic units of $\mathbb{Q}(\zeta_{p^{n+1}})$ congruent to 1 modulo $1 - \zeta_{p^{n+1}}$, and let $C = \varprojlim C_n$ with respect to the norm maps. Recall that $e_\chi C_n$ is generated by

$$\left(\zeta_{p^{n+1}}^{(1-\delta_0\gamma_0)/2} \frac{\zeta_{p^{n+1}}^{\delta_0\gamma_0} - 1}{\zeta_{p^{n+1}} - 1} \right)^{(p-1)e_\chi}$$

where δ_0 generates Δ . The term above defines a norm coherent sequence in $e_\chi C$ and we let ξ_χ denote the associated distribution. What's more, there exists a $p-1$ -st root of unity $\zeta_\chi \in \mathbb{Z}_p$ not equal to 1 such that $e_\chi U_n$ is generated by

$$\left(\frac{\zeta_\chi - \zeta_{p^{n+1}}}{\zeta} \right)^{e_\chi}$$

where ζ is the $p-1$ -st root of unity such that $\zeta_\chi - 1 \equiv \zeta \pmod{\zeta_{p^{n+1}} - 1}$ (see [?, Theorem 13.54]). Once again, the term above forms a norm coherent sequence in $e_\chi U$ and we let v_χ denote the associated distribution. From Theorem 4.1, we have

$$\widehat{v}_\chi(\zeta_\psi) = - \sum_{b=1}^{f_\varphi} \log_p (\zeta_\chi - \zeta_{f_\varphi}^b) \overline{\varphi(b)}$$

where $\varphi = \chi\psi$. Since the terms $\log_p(\zeta_\chi - \zeta_{f_\varphi}^b)$ for $(b, f_\varphi) = 1$ are linearly independent over \mathbb{Q} , they must also be linearly independent over \mathbb{Q}^{alg} by a theorem of Brumer. Hence $\widehat{v}_\chi(\zeta_\psi) \neq 0$.

Now, there exists a distribution $\mu_\chi \in \mathcal{M}(\Gamma)$ such that $\xi_\chi = \mu_\chi * v_\chi$. Specifically, μ_χ is formed from the coherent sequence of group ring elements in $\varprojlim \mathbb{Z}_p[[\Gamma_n]]$ that map the generator for $e_\chi U_n$ to the generator for $e_\chi C_n$. There exists $H(T) \in \Lambda^\times$ such that

$$\widehat{\mu}_\chi(T) \cdot H(T) = G_\chi(T) \in \Lambda$$

where

$$G_\chi((1+p)^s) = L_p(1-s, \chi),$$

(see [?] or [?, Theorem 13.56]). Let $F_\chi(T) \in \Lambda$ such that $L_p(s, \chi\psi) = F_\chi(\zeta_\psi(1+p)^s)$. Then F_χ and G_χ are related via the formula

$$F_\chi(T) = G_\chi\left(\frac{1+p}{T}\right).$$

Since $\widehat{v}_\chi(\zeta_\psi) \neq 0$, by Proposition 3.2 we have that

$$\left[\widehat{\mu}_\chi(T) = \frac{\widehat{\xi}_\chi(T)}{\widehat{v}_\chi(T)}\right]_{T=\zeta}$$

for any p^n -th root of unity ζ . It follows that

$$L_p(1, \chi\psi) = F_\chi(\zeta_\psi(1+p)) = G_\chi(\zeta_\psi) = \frac{\widehat{\xi}_\chi(\zeta_\psi)}{\widehat{v}_\chi(\zeta_\psi)} \cdot H(\zeta_\psi).$$

On the other hand, by Corollary 4.2 we have

$$L_p(1, \chi\overline{\psi}) = \frac{\widehat{\xi}_\chi(\zeta_\psi)}{\tau(\overline{\chi\psi})(\chi(\delta_0)\zeta_\psi - 1)}.$$

Dividing the formula for $L_p(1, \chi\overline{\psi})$ by the formula for $L_p(1, \chi\psi)$ yields

$$\frac{L_p(1, \chi\overline{\psi})}{L_p(1, \chi\psi)} = \frac{\widehat{v}_\chi(\zeta_\psi)}{\tau(\overline{\chi\psi})(\chi(\delta_0)\zeta_\psi - 1)H(\zeta_\psi)}$$

whence

$$\widehat{v}_\chi(\zeta_\psi) = \tau(\overline{\chi\psi}) \left[(\chi(\delta_0)\zeta_\psi - 1)H(\zeta_\psi)F_\chi(\zeta_\psi(1+p))^{1-\sigma} \right]$$

where $\sigma \in \text{Gal}(\mathbb{Q}_p(\zeta_\psi)/\mathbb{Q}_p)$ is defined by $\iota : \zeta_\psi \mapsto \zeta_\psi^\sigma$. Since χ is non-trivial, the term above in brackets is a unit of $\mathbb{Z}_p[\zeta_\psi]$. \square

Keeping notation from the proof of Theorem 4.3, let $R_p(e_\chi U_n)$ denote the p -adic regulator of $e_\chi U_n$. Specifically, for any set of elements $x_1, x_2, \dots, x_{p^n} \in e_\chi U_n$ that generate $e_\chi U_n$ as a \mathbb{Z}_p -module, set

$$R_p(e_\chi U_n) = \det(\log_p(x_j^\gamma))_{j,\gamma}$$

where γ ranges over Γ_n . Note that $R_p(e_\chi U_n)$ is determined only up to a unit of \mathbb{Z}_p .

Corollary 4.4. *There exists $(y_n) \in U$ such that $(y_n^{e_\chi})$ generates $e_\chi U$ and the associated distribution v_χ satisfies*

$$\prod_{\psi \in \widehat{\Gamma}_n} \widehat{v}_\chi(\zeta_\psi) = \prod_{\psi \in \widehat{\Gamma}_n} \tau(\overline{\chi\psi}) = R_p(e_\chi U_n).$$

Proof. Let $(\mu_n) \in \mathbb{Z}_p[[\Gamma]]$ be associate to the power series

$$\left[\left(\frac{\chi(\delta_0)}{T} - 1 \right) H(T) \right]^{-1} \in \Lambda^\times$$

so that

$$y_n^{e_\chi} = \left(\frac{\zeta_\chi - \zeta_{p^{n+1}}}{\zeta} \right)^{\mu_n e_\chi}$$

also generates $e_\chi U_n$. Let $v_\chi \in \mathcal{K}(\Gamma)$ be associate to the sequence $(y_n) \in U$. Note that $\widehat{v}_\chi(\zeta_\psi)$ is the $\chi\psi$ -part of $R_p(e_\chi U_n)$. To be precise, let

$$\Upsilon^{(n)} := - \sum_{\delta \in \Delta} \sum_{a=0}^{p^n-1} \log_p(y_n^{\gamma_0^a \delta}) (\delta \gamma_0^a)^{-1} \in \mathbb{Q}_p(\zeta_{p^{n+1}})[\Delta \times \Gamma_n].$$

Now let $\Upsilon_{\chi\psi}^{(n)} \in \mathbb{Q}_p(\zeta_{p^{n+1}})$ be defined by $e_{\chi\psi} \Upsilon^{(n)} = \Upsilon_{\chi\psi}^{(n)} e_{\chi\psi}$. Then the regulator for $e_\chi U_n$ is the product

$$\prod_{\psi \in \widehat{\Gamma}_n} \Upsilon_{\chi\psi}^{(n)} = \prod_{\psi \in \widehat{\Gamma}_n} \widehat{v}_\chi(\zeta_\psi) = \prod_{\psi \in \widehat{\Gamma}_n} \tau(\overline{\chi\psi}) \cdot F_\chi(\zeta_\psi(1+p))^{1-\iota} = \prod_{\psi \in \widehat{\Gamma}_n} \tau(\overline{\chi\psi}).$$

□

Under the additional assumption that p is regular, the above equality of products can be refined into an equality of components. In particular, keeping notation from Theorem 4.3 and Corollary 4.4, we have

Corollary 4.5. *If p is a regular prime, then there exists $(v_n) \in U$ such that $(v_n^{e_\chi})$ generates $e_\chi U$ and the associated distribution ν_χ satisfies*

$$\widehat{\nu}_\chi(\zeta_\psi) = \tau(\overline{\chi\psi}) = N_{\chi\psi}^{(n)},$$

where $N_{\chi\psi}^{(n)}$ is to (v_n) as $\Upsilon_{\chi\psi}^{(n)}$ is to (y_n) .

Proof. If p is a regular prime, then $F_\chi(T) \in \Lambda^\times$, hence

$$G_\chi(T), G_\chi\left(\frac{1+p}{T}\right) \in \Lambda^\times$$

as well. So the values $F_\chi(\zeta_\psi(1+p))^{1-\sigma}$ are interpolated by a power series in Λ^\times , namely,

$$F_\chi(\zeta_\psi(1+p))^{1-\sigma} = \frac{G_\chi(T)}{G_\chi((1+p)/T)} \Big|_{T=\zeta_\psi}.$$

Let $\mu_n \in \mathbb{Z}_p[[\Gamma]]$ be associate to the power series

$$\left[\frac{G_\chi(T)}{G_\chi((1+p)/T)} \right]^{-1}$$

so that

$$v_n^{e_\chi} = y_n^{\mu_n e_\chi}$$

also generates $e_\chi U_n$. Let ν_χ be associate to the sequence (v_n) . It follows that

$$N_{\chi\psi}^{(n)} = - \sum_{a=0}^{p^n-1} \overline{\psi}(a) \sum_{\delta \in \Delta} \log_p(v_n^{\gamma_0^a \delta}) \overline{\chi}(\delta) = \widehat{\nu}_\chi(\zeta_\psi) = \tau(\overline{\chi\psi}).$$

□

Remark 4.6. Note that $\mu_\chi \in \mathcal{M}(\Gamma)$ from the proof of Theorem 4.3 can be given explicitly in terms of $\widehat{v}_\chi(T)$ and $\widehat{\xi}_\chi(T)$ from Theorem 4.3. In particular,

$$\mu_\chi(a + p^n \mathbb{Z}_p) = \frac{1}{p^n} \sum_{c=0}^{p^n-1} \zeta_{p^n}^{-ca} \left. \frac{\widehat{\xi}_\chi(T)}{\widehat{v}_\chi(T)} \right|_{T=\zeta_{p^n}^c}.$$

Moreover, let E_n denote the units of $\mathbb{Q}(\zeta_{p^{n+1}})$. The map $E_n \rightarrow \mathbb{Z}_p[\Gamma_n]$ defined by

$$\epsilon \mapsto (\Upsilon_\chi^{(n)})^{-1} \cdot \sum_{a=0}^{p^n-1} \log_p(\epsilon^{(p-1)e_\chi \gamma_0^a}) \gamma_0^{-a}$$

is a $\text{Gal}(\mathbb{Q}(\zeta_{p^{n+1}})/\mathbb{Q})$ -module map where the invertibility of $\Upsilon_\chi^{(n)}$ follows from the non-vanishing of $\widehat{v}_\chi(\zeta_\psi)$. Note that $(\Upsilon_\chi^{(n)})^{-1}$ is acting as an *integralizer* in the sense of [?, Definition 2.5]. The image of the cyclotomic units of $\mathbb{Q}(\zeta_{p^{n+1}})$ under this map annihilates the χ -part of the Sylow p -subgroup of $\text{Cl}(\mathbb{Q}(\zeta_{p^{n+1}}^+))$ [?, Theorem 3.1]. Let $M_\chi^{(n)} \in \mathbb{Z}_p[\Gamma_n]$ denote the group ring element:

$$M_\chi^{(n)} = \sum_{a=0}^{p^n-1} \mu_\chi(a + p^n \mathbb{Z}_p) \gamma_0^{-a} \in \mathbb{Z}_p[\Gamma_n].$$

Likewise, let $\Upsilon_\chi^{(n)}$ and $\Xi_\chi^{(n)}$ be the group ring elements in $\mathbb{Q}_p(\zeta_{p^{n+1}})[\Gamma_n]$ corresponding to the distributions v_χ and ξ_χ , respectively. Since

$$M_\chi^{(n)} = (\Upsilon_\chi^{(n)})^{-1} \cdot \Xi_\chi^{(n)}$$

we see that $M_\chi^{(n)}$ is indeed in the image of the cyclotomic units of the aforementioned map. Therefore $(M_\chi^{(n)}) \in \mathbb{Z}_p[[\Gamma]]$ is a coherent sequence of explicit annihilators of the χ -part of the Sylow p -subgroup of $\text{Cl}(\mathbb{Q}(\zeta_{p^{n+1}}^+))$.

E-mail address: timothy.all@rose-hulman.edu

5500 WABASH AVE, TERRE HAUTE, IN 47803

E-mail address: waller@math.osu.edu

231 W 18TH AVE, COLUMBUS, OH 43210